Midterm 3

Math 347, Fall 2015

Section G1

Last (Family) Name: ____________________

First (Given) Name: ____________________

NetID: ____________________

Instructions

All cell phones, calculators, and other devices must be turned off and out of reach. Also, all books, notebooks, and scratch papers must be out of reach. The last page of the test is scratch paper, use it if needed, but it will not be graded.

Problem	Score
1	/ 11
2	/ 13
3	/ 16
Total	/ 40

1. (11 points)

(a) Complete the following definitions:

(i) (2 points) A real $u \in \mathbb{R}$ is said to be an *upper bound* for a set $A \subseteq \mathbb{R}$ if $\forall a \in A\ a \leq u$.

(ii) (2 points) A real $u \in \mathbb{R}$ is called the *supremum* (or, the *least upper bound*) of a set $A \subseteq \mathbb{R}$ if u is an upper bound and for any other upper bound v for A, $u \leq v$. The last condition is equivalent to saying that any $v < u$ is not an upper bound for A.

(b) Let $A, B \subseteq \mathbb{R}$ be nonempty sets such that $\forall a \in A\ \forall b \in B\ a \leq b$; in words, every element of A is less than or equal to every element of B (draw a picture).

(i) (1 point) Is every element of B an upper bound for A? Just write YES or NO here: YES

(ii) (3 points) Deduce that $\sup(A)$ is a lower bound for B.

Solution. We need to show that $\forall b \in B$, $\sup(A) \leq b$. Fix an arbitrary $b \in B$. By the previous part, b is an upper bound for A, so, by the definition of $\sup(A)$, $\sup(A) \leq b$.

(iii) (3 points) Conclude that $\sup(A) \leq \inf(B)$.

Solution. By the previous part, $\sup(A)$ is a lower bound for B, so it must be $\leq \inf(B)$, by the very definition of $\inf(B)$.

2. (13 points)

(a) Let $(x_n)_n$ be a sequence and $L \in \mathbb{R}$.

(i) (2 points) Write the definition of $\lim_{n\to\infty} x_n = L$ using the term **eventually**.

Definition. $\lim_{n\to\infty} x_n = L$ if for every $\varepsilon > 0$ **eventually** $|x_n - L| < \varepsilon$.

(ii) (2 points) Write the definition of $\lim_{n\to\infty} x_n = L$ **without** using the term **eventually**.

Definition. $\lim_{n\to\infty} x_n = L$ if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall n \geq N$, $|x_n - L| < \varepsilon$.

(b) (4 points) Let $(x_n)_n$ and $(y_n)_n$ be sequences and suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Using only the definition of limit (with or without the term **eventually**), show that $\lim_{n\to\infty}(y_n - x_n) = y - x$.

Solution. Fix an arbitrary $\varepsilon > 0$. We need to show that **eventually** $|y_n - x_n - (y - x)| < \varepsilon$. By the triangle inequality, we have

$$|y_n - x_n - (y - x)| = |(y_n - y) + (x - x_n)| \leq |y_n - y| + |x - x_n|,$$

and because $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, we know that **eventually** $|x - x_n| < \frac{\varepsilon}{2}$ and **eventually** $|y_n - y| < \frac{\varepsilon}{2}$. It follows that **eventually** both $|x - x_n| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$ hold, so **eventually**,

$$|y_n - x_n - (y - x)| \leq |y_n - y| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(c) (5 points) Let $(a_n)_n$ be an increasing sequence and $(b_n)_n$ be a decreasing sequence. Suppose that $\forall n, m \in \mathbb{N}$ $a_n \leq b_m$ and $\lim_{n\to\infty}(b_n - a_n) = 0$. Show that both of $(a_n)_n$ and $(b_n)_n$ converge, and in fact, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. You may use statements proved in class.

Hint: Once you prove that $(a_n)_n$ and $(b_n)_n$ converge, it may help to denote their limits by a and b, respectively.

Solution. Because $\forall n \in \mathbb{N}$ $a_n \leq b_1$ and $\forall m \in \mathbb{N}$ $a_1 \leq b_m$, b_1 is an upper bound for $(a_n)_n$ and a_1 is a lower bound for $(b_n)_n$, so by the Monotone Convergence Theorem, both $(a_n)_n$ and $(b_n)_n$ converge. Let a and b denote their limits, respectively. Then by the previous part, we know that $\lim_{n\to\infty}(b_n - a_n) = b - a$. Thus, by the hypothesis, $b - a = 0$, so $a = b$.

3. (16 points) Determine whether the following statements are true or false, and prove your answers. You may use statements proved in class, but when showing convergence/divergence, do **not** use the term **eventually** (it will not help anyway).

(a) (4 points) $\left(\frac{4n^2-1}{n+2}\right)_n$ converges.

Solution. FALSE. $\frac{4n^2-1}{n+2} = \frac{4n-\frac{1}{n}}{1+\frac{2}{n}} \geq 4n - \frac{1}{n} \geq 3n \geq n$ and the sequence $(n)_n$ is unbounded, so our sequence is unbounded as well, and thus, diverges.

(b) (4 points) $\left(\frac{4n-1}{n+2}\right)_n$ converges.

Solution. TRUE. $\frac{4n-1}{n+2} = \frac{4-\frac{1}{n}}{\frac{1}{n}+\frac{2}{n}}$. We use the theorem about limits and algebraic operations, working from the bottom up:

(i) Basics: As shown in class, $\frac{1}{n} \to 0$. Also, any constant sequence converges to that constant.

(ii) Product of limits: Thus, $\frac{2}{n} \to 0$ because $\frac{2}{n} = 2 \cdot \frac{1}{n} \to 2 \cdot 0$. Similarly,

(iii) Sum of limits: $4 - \frac{1}{n} \to 4 - 0 = 4$ and $1 + \frac{2}{n} \to 1 + 0 = 1$.

(iv) Ratio of limits: $\frac{4-\frac{1}{n}}{1+\frac{2}{n}} \to \frac{4}{1} = 4$.

(c) (4 points) $\sin(\frac{\pi n}{4}) \to 0$.

Solution. FALSE. Intuitively, the reason is that the values ± 1 keep appearing in the sequence no matter how far we go. Formally, we need to show that $\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists n \geq N \, |\sin(\frac{\pi n}{4})| \geq \varepsilon$. Taking $\varepsilon := 1$ works. Indeed, fix an arbitrary $N \in \mathbb{N}$. We now have to find $n \geq N$ such that $|\sin(\frac{\pi n}{4})| \geq 1$. But this is easy, just take n to be any number bigger than N that is of the form $4k+2$ (i.e. even, but not divisible by 4). Then $|\sin(\frac{\pi n}{4})| = |\sin(\frac{\pi(4k+2)}{4})| = |\sin(k\pi + \frac{\pi}{2})| = |\pm 1| = 1$.

(d) (4 points) $\frac{\sin(\frac{\pi n}{4})}{n} \to 0$.

Solution 1. TRUE. We showed in class that if $x_n \to 0$ and $(y_n)_n$ is bounded, then $x_n y_n \to 0$. In our case, $x_n = \frac{1}{n}$ and $y_n = \sin(\frac{\pi n}{4})$. Our $(y_n)_n$ is bounded because for every $n \in \mathbb{N}$, $|y_n| = |\sin(\frac{\pi n}{4})| \leq 1$. Our $x_n \to 0$. Thus $\frac{\sin(\frac{\pi n}{4})}{n} = x_n y_n \to 0$.

Solution 2. TRUE. $0 \leq |\frac{\sin(\frac{\pi n}{4})}{n}| \leq \frac{1}{n}$, so by the Squeeze Theorem, $|\frac{\sin(\frac{\pi n}{4})}{n}| \to 0$ and hence also $\frac{\sin(\frac{\pi n}{4})}{n} \to 0$.

Solution 3. TRUE. We show this directly from the definition. Fix $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ such that $\forall n \geq N$, $|\frac{\sin(\frac{\pi n}{4})}{n}| < \varepsilon$. But for every $n \in \mathbb{N}$,

$$\left|\frac{\sin(\frac{\pi n}{4})}{n}\right| = \frac{|\sin(\frac{\pi n}{4})|}{n} \leq \frac{1}{n},$$

so it is enough to find $N \in \mathbb{N}$ such that $\forall n \geq N$, $\frac{1}{n} < \varepsilon$. Take N to be any natural number greater than $\frac{1}{\varepsilon}$, for example $N := \lceil \frac{1}{\varepsilon} \rceil + 1$. Then, for any $n \geq N$, we have $\frac{1}{n} \geq \frac{1}{N} > \varepsilon$. Thus, we have that for every $n \geq N$,

$$\left|\frac{\sin(\frac{\pi n}{4})}{n}\right| \leq \frac{1}{n} < \varepsilon.$$

Scratch paper